STUDENT'S NAME: ____________________ ___ ________ ___

BAULKHAM HILLS HIGH SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

2004

MATHEMATICS

EXTENSION 2

Time Allowed - Three hours

GENERAL INSTRUCTIONS:

- Reading time - 5 minutes.
- Working time - 3 hours.
- Write using blue or black pen.
- Start each question on a new page.
- Write your Student Number at the top of each page.
- Calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every Question.

QUESTION 1

(a) Use the substitution $u = x^2$ to calculate.

$$\int_0^{\frac{1}{\sqrt{2}}} \frac{x\,dx}{\sqrt{1 - x^4}}\ .$$

(b) Find: $\int \sec^4 x \, dx.$

(c) By rationalising the numerator, show that:

Show that $$\int_0^1 \sqrt{\frac{1+x}{3+x}}\,dx = \sqrt{8} - \sqrt{3} + \ln\left(\frac{2+\sqrt{3}}{3+\sqrt{8}}\right).$$

(d) Find real numbers A, B and C such that

$$\frac{4x + 2}{(x + 3)(x^2 + 1)} \equiv \frac{A}{x + 3} + \frac{Bx + C}{x^2 + 1}$$ and hence find:

$$\int \frac{4x + 2}{(x+3)(x^2+1)}\,dx$$

(e) Use integration by parts to find $\int x \tan^{-1} x \, dx$

QUESTION 2.

(a) let $z = 1 + 2i$ and $\omega = 2 - i$. Find in the form $x + iy$.

(i) $z\,\overline{\omega}$.

(ii) $\dfrac{1}{\omega^2}$.

(b) Sketch the region in the complex plane where the inequalities

$|z - 3 - 4i| \le 5$ and $0 \le \arg z \le \dfrac{\pi}{4}$ both hold.

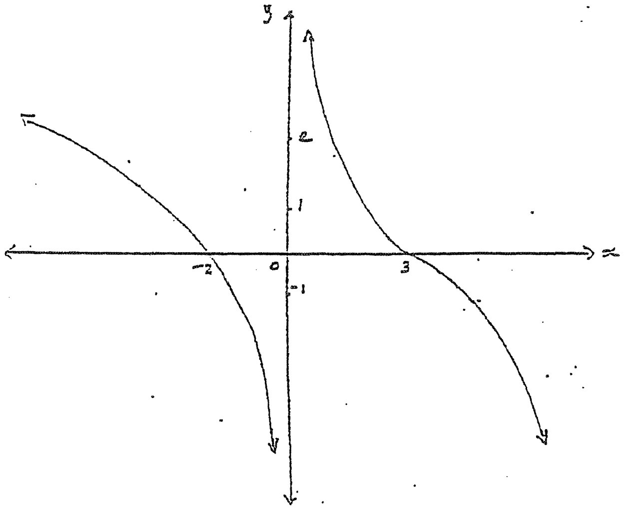
(c) Given that $1 - i$ is a root of $x^3 - 3x^2 + 4x - 2 = 0$, find the other two roots.

(d) If $\sqrt{3} + i$ and two other complex numbers form the vertices of an equilateral triangle on the complex plane with its centre at the origin, find the two other complex numbers.

(e) Express $(-1 + i)$ in modulus argument form and hence evaluate $(-1 + i)^8$.

QUESTION 3

(a) The diagram shows the graph of $y = f(x)$.



Draw separate half page sketches of the graphs of the following showing all obvious features.

(i) $y = \dfrac{1}{f(x)}$.

(ii) $y = \sqrt{f(x)}$.

(iii) $y = f(|x|)$

(iv) $y = \ln f(x)$.

(b) An ellipse centred at the origin has a focus at $(-4, 0)$ and a directrix $x = 9$, find its equation.

(c) The curve $y = \dfrac{ax^2 + bx + c}{cx^2 + bx + a}$ has a horizontal asymptote of $y = 4$. Find where this curve cuts the y axis.

(d) If z lies on the locus of $|z - 2| = 2$ in the Argand diagram, show that:
$|z|^2 + |z - 4|^2$ is a constant

QUESTION 4

(a) The area bounded by the curve $y = 12x - x^2$, the x axis, $x = 2$ and $x = 10$ is rotated about the y axis to form a solid By using the method of cylindrical shells calculate the volume of the solid.

(b) The base of a solid is the segment of the parabola $x^2 = 4y$ cut off by the line $y = 2$. Each cross section perpendicular to the y axis is a right angled isosceles triangle with hypotenuse in the base of the solid. Find the volume of the solid.

(c) The normal at a point $P(cp, \frac{c}{p})$ on the rectangular hyperbola $xy = c^2$ meets the x axis at Q. M is the mid point of PQ.

(i) By proving that the equation of this normal is:

$$y - \frac{c}{p} = p^2(x - cp)$$

show that M is the point $\left(\frac{2cp^4 - c}{2p^3}, \frac{c}{2p}\right)$.

(ii) Hence show that the locus of M is given by:

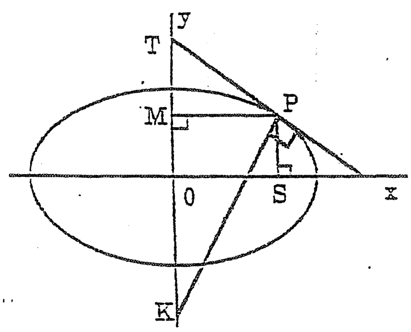
$$2xyc^2 = c^4 - 8y^4.$$

QUESTION 5

(a) The diagram shows the point $P\left(a\cos\theta, b\sin\theta\right)$ on the ellipse.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

P lies vertically above S, one of the foci of the ellipse.The tangent at P meets the y-axis at T and the normal at P meets the y-axis at K. M is the foot of the perpendicular from P to the y-axis.



(i) Write down the co-ordinates of S.

(ii) Show that $\sin\theta=\sqrt{1-e^2}$ where e is the eccentricity of the ellipse

(iii) Find the co-ordinates of T,M and K. You may assume that the equation of the tangent at P is : $y-b\sin\theta=\dfrac{-b\cos\theta}{a\sin\theta}\left(x-a\cos\theta\right)$

(iv) Prove that the area of triangle TPK is:

$$A=\frac{abe}{2}\times\frac{(e^2+1)}{\sqrt{1-e^2}}$$

(b) The cubic $y = x^3 - px + q$ has two turning points

(i) Show that $p > o$.
(ii) Find the co-ordinates of these turning points.
(iii) The line $y = k$ meets the cubic in 3 points.

Show that $$q-\frac{2p}{3}\sqrt{\frac{p}{3}} < k < q+\frac{2p}{3}\sqrt{\frac{p}{3}}.$$

(c) Sketch without using calculus

$$y=\frac{2\left(x^2-8x\right)}{x^2+x-20}.$$

QUESTION 6

(a) When $p(x) = x^4 + ax^2 + bx$ is divided by $x^2 + 1$ the remainder is $x + 2$. Find the values of a and b.

(b) Given that the equation $ax^4 + 4bx + c = 0$ has a double root, prove that

$$ac^3 = 27b^4.$$

(c) The cubic equation $x^3 + px + q = 0$ has 3 real non zero roots α, β, χ Find in terms of the constants p and q the value of:

(i) $\alpha^2 + \beta^2 + \chi^2$.

(ii) $(\alpha - 1)(\beta - 1)(\chi - 1)$.

(iii) $\alpha^3 + \beta^3 + \chi^3$.

(iv) $\alpha^4 + \beta^4 + \chi^4$

QUESTION 7

(a) A particle of mass m is projected downwards under gravity, in a medium whose resistance is equal to the velocity of the particle multiplied by $\frac{mg}{T}$. Show that the terminal velocity of this particle is T.

If a particle is projected vertically upwards in the same medium with velocity u, show that it attains a height of:

$$H = \frac{uT}{g} + \frac{T^2}{g}\log\left(\frac{T}{u+T}\right).$$

(b) For what rational value (s) of k do the two equations:

$$2x^2 - 7x + k = 0 \quad \text{and}$$

$$2x^3 - x^2 - 37x + 36 = 0,$$

have a common root?

(c) Show that $(1+x)^n\left(1+\frac{1}{x}\right)^n = \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^{2n}$ and hence prove that

$$\sum_{r=0}^{n} \left({}^nC_r\right)^2 = {}^{2n}C_n.$$

QUESTION 8

(a) By using De Moivre's Theorem prove that

$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ and hence solve the equation:

$16x^5 - 20x^3 + 5x = 0$

and deduce the values of

(i) $\cos\frac{\pi}{10}$.

(ii) $\cos\frac{3\pi}{10}$.

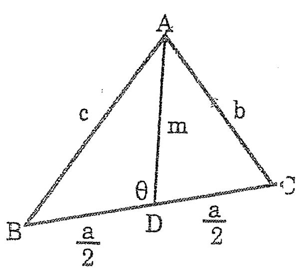
(b) Given that $I_n = \int_0^{\frac{\pi}{2}} x^n \sin x \, dx$

Prove that $I_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}$ and hence evaluate:

$\int_0^{\frac{\pi}{2}} x^4 \sin x \, dx$

(c) In ΔABC a median is drawn from A to meet BC at D. Prove that the length of the median is given by:

$$m = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$$



STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \ \textit{if} \ n < 0$$

$$\int \frac{1}{x} dx = \ln x, \qquad x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a}, \ a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln(x + \sqrt{x^2 - a^2}), \ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln(x + \sqrt{x^2 + a^2})$$

NOTE: $\ln x = \log_e x, \quad x > 0$

BAULKHAM HILLS HIGH
TRIAL EXT 2 2004 SOLUTIONS

1 a) $\int_0^{\frac{1}{\sqrt2}} \frac{x\,dx}{\sqrt{1-x^4}}$

let $u = x^2$ $\quad x=\frac{1}{\sqrt2}$ $u=\frac12$

$du = 2x\,dx$ ✓ $\quad x=0$ $u=0$

$\therefore I = \frac12\int_0^{\frac12}\frac{du}{\sqrt{1-u^2}}$

$= \frac12\left[\sin^{-1}u\right]_0^{\frac12}$ ✓

$= \frac12\left[\frac{\pi}{6}-0\right]$

$= \frac{\pi}{12}$ ✓ (3)

b) $\int \sec^4 x\,dx$

$= \int (1+\tan^2 x)\sec^2 x\,dx$

$= \int \sec^2 x\,dx + \int \tan^2 x\sec^2 x\,dx$ ✓

$= \tan x + \frac13\tan^3 x + C$ ✓ (2)

c) $\int_0^1 \sqrt{\frac{1+x}{3+x}} \times \frac{\sqrt{1+x}}{\sqrt{1+x}}\,dx$

$= \int_0^1 \frac{1+x}{\sqrt{3+4x+x^2}}\,dx$ ✓

$= \int_0^1 \frac{\frac12(2x+4)-1}{\sqrt{(x+2)^2-1}}\,dx$ ✓

$= \left[\sqrt{x^2+4x+3}\right]_0^1$ ✓ $- \left[\ln\left(x+2+\sqrt{x^2+4x+3}\right)\right]_0^1$ ✓

$= \sqrt8-\sqrt3-\left[\ln(3+\sqrt8)-\ln(2+\sqrt3)\right]$

$= \sqrt8-\sqrt3+\ln(2+\sqrt3)-\ln(3+\sqrt8)$

$= \sqrt8-\sqrt3+\ln\left(\frac{2+\sqrt3}{3+\sqrt8}\right)$ ✓ (5)

d) $A(x^2+1)+(x+3)(Bx+C) \equiv 4x+2$

subst $x=-3$

$\therefore 10A = -10 \Rightarrow A=-1$

const term $A+3C=2$

$\therefore 3C = 3 \Rightarrow C=1$

term in x^2 $\quad A+B=0$

$\therefore B=1$

$A=-1$, $B=1$, $C=1$ (3)

$\therefore \int\frac{4x+2}{(x+3)(x^2+1)}\,dx = \int\frac{x+1}{x^2+1}\,dx - \int\frac{dx}{x+3}$

$= \frac12\int\frac{2x\,dx}{x^2+1} + \int\frac{dx}{x^2+1} - \int\frac{dx}{x+3}$ ✓

$= \frac12\ln(x^2+1)+\tan^{-1}x + \ln(x+3)+C$ (4)

e) $I = \int x\tan^{-1}x\,dx$

let $u=\tan^{-1}x$ $\quad v'=x$

$u'=\frac{1}{1+x^2}$ $\quad v=\frac{x^2}{2}$ ✓

$\therefore I = \frac{x^2}{2}\tan^{-1}x - \frac12\int\frac{x^2}{1+x^2}\,dx$ ✓

$= \frac{x^2}{2}\tan^{-1}x - \frac12\int\frac{x^2+1-1}{1+x^2}\,dx$ (3)

$= \frac{x^2}{2}\tan^{-1}x - \frac{x}{2} + \frac12\tan^{-1}x + C$ ✓

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2 a) i) $z\bar{w} = (1+2i)(2+i)$ ✓

$= (2-2)+(4+1)i$

$= 5i$ ✓

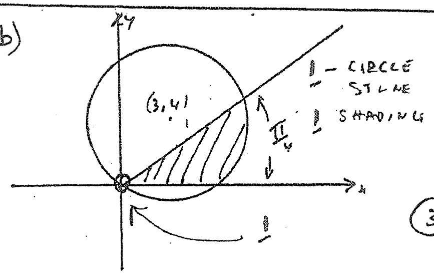
ii) $\frac{1}{w^2} = \frac{1}{(2-i)^2}$

$= \frac{1}{3-4i}\times\frac{3+4i}{3+4i}$ ✓

$= \frac{3+4i}{25}$

$= \frac{3}{25}+\frac{4}{25}i$ ✓ (4)

2b)



(3)

c) if $(1-i)$ is a root so is $(1+i)$ ✓

$\therefore P(x) = (x-\alpha)(x-(1-i))(x-(1+i))$

$= (x-\alpha)(x^2-2x+2)$ ✓ (3)

$-2\alpha = -2$

$\therefore \alpha = +1$

$\therefore$ other two roots are $(1+i)$ and 1 ✓

d) if $z_1 = \sqrt3+i$

$z_2 = (\sqrt3+i)\,\text{cis}\frac{2\pi}{3}$

$= \left(-\frac{\sqrt3}{2}-\frac{\sqrt3}{2}\right)+i\left(\frac32-\frac12\right)$

$= -\sqrt3+i$ ✓

$z_3 = (\sqrt3+i)\,\text{cis}\left(-\frac{2\pi}{3}\right)$

$= \left(-\frac{\sqrt3}{2}+\frac{\sqrt3}{2}\right)+i\left(-\frac12-\frac32\right)$ (2)

$= 0 - 2i$ ✓

$\therefore$ other 2 complex nos are $i-\sqrt3$, $-2i$.

e) $(-1+i) = \sqrt2\left(-\frac{1}{\sqrt2}+\frac{i}{\sqrt2}\right)$

$= \sqrt2\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$ ✓

or $= \sqrt2\,\text{cis}\frac{3\pi}{4}$.

$(-1+i)^8 = (\sqrt2)^8\,\text{cis}\,\frac{3\pi}{4}\times 8$ ✓

$= 16(\cos 0 + i\sin 0)$

$= 16$ ✓ (3)

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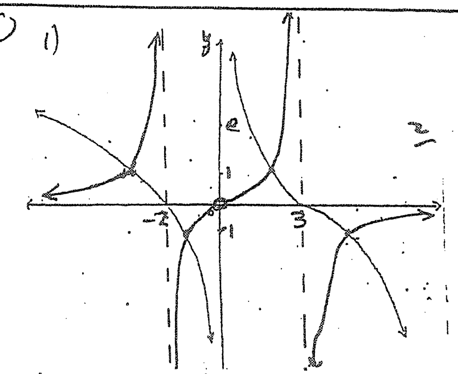
3 i)



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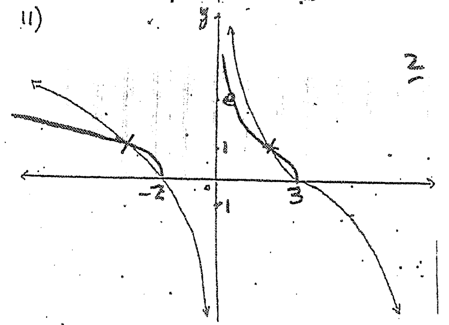
ii)



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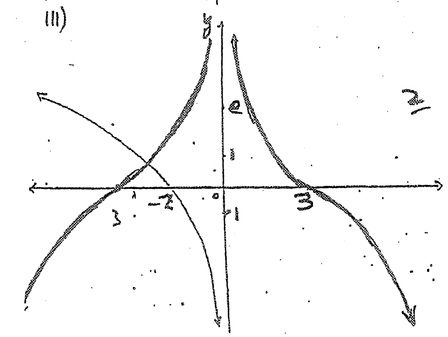
iii)



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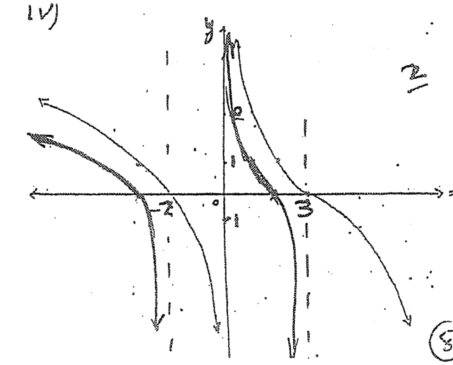
iv)



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(8)

3b)

$ae = 4$

$\frac{a}{e} = 9$

$\therefore a^2 = 36$ 1

$a = 6 \quad e = \frac{2}{3}$ 1

$b^2 = a^2(1-e^2)$

$b^2 = 36\left(1 - \frac{4}{9}\right)$

$= 36 \cdot \frac{5}{9}$

$b = \sqrt{20}$ 1

$\therefore$ ellipse is $\frac{x^2}{36} + \frac{y^2}{20} = 1$ (3)

c) as $x \to \infty$

$y \to \frac{a}{c} = 4$ 1

cuts y axis when $x = 0$

ie $y = \frac{c}{a}$

$= \frac{1}{4}$ 1 (2)

d) let $z = x + iy$

$\therefore |z-2|^2 = (x-2)^2 + y^2 = 4$

$\therefore x^2 - 4x + y^2 = 0$ 1

$|z|^2 + |z-4|^2 = x^2 + y^2 + (x-4)^2 + y^2 + 16$

$= x^2 + y^2 + x^2 - 8x + 16 + y^2 + 16$

$= 2(x^2 - 4x + y^2) + 16$

$= 16$ 1 (2)

$\therefore |z|^2 + |z-4|^2 = 16$ ie a Centre at

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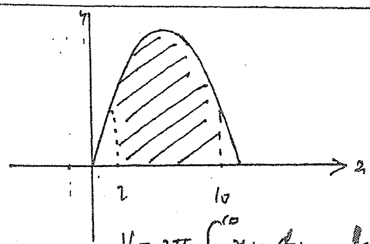
④ a)



$V = 2\pi \int_2^{10} xy \, dx$ 1

$= 2\pi \int_2^{10} x(12x - x^2) \, dx$ 1

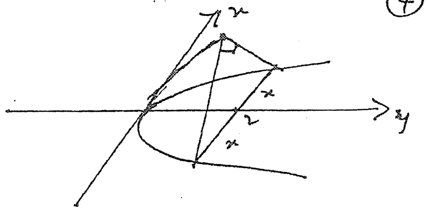
$= 2\pi \int_2^{10} 12x^2 - x^3 \, dx$

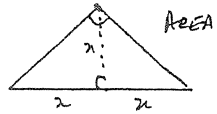
$= 2\pi \left[4x^3 - \frac{x^4}{4}\right]_2^{10}$ 1

$= 2\pi[4000 - 2500] - 2\pi[32 - 4]$

$= 2944\pi$ cc units 1 (4)

b)





AREA $\triangle = \frac{1}{2} \times x \times 2x$

$= x^2$ 1

$\delta V = x^2 \delta y$

$\therefore V = \int_0^2 x^2 \, dy$

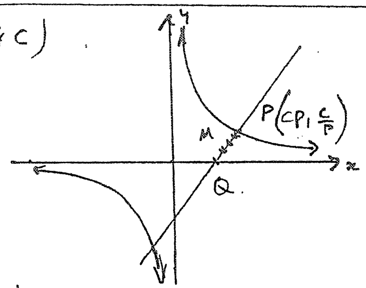
$= \int_0^2 4y \, dy$ 1

$= \left[2y^2\right]_0^2$

$= 8$ cu units 1

(3)

4c)



i) $xy = c^2$

$y = \frac{c^2}{x}$

$y' = -\frac{c^2}{x^2}$ 1

subst $x = cp \quad \therefore y' = -\frac{1}{p^2}$

$\therefore$ grad of normal $= p^2$ 1

equat of N is $y - \frac{c}{p} = p^2(x - cp)$ *

subst $y = 0$ to find Q

$\therefore -\frac{c}{p} = p^2(x - cp)$

$x - cp = -\frac{c}{p^3}$

$x = \left(cp - \frac{c}{p^3}\right)$

$\therefore$ Q is pt $\left(cp - \frac{c}{p^3}, 0\right)$ 1

M is pt $\left(\frac{\left(cp - \frac{c}{p^3}\right) + cp}{2}, \frac{c}{2p}\right)$

$= \left(\frac{2cp^4 - c}{2p^3}, \frac{c}{2p}\right)$ 1

ii) $2xyc^2 = 2\left(\frac{2cp^4 - c}{2p^3}\right) \cdot \frac{c}{2p} \cdot c^2$

$= \frac{2c^4p^4 - c^4}{2p^4}$ 1 *

$c^4 - 8y^4 = c^4 - 8 \times \left(\frac{c}{2p}\right)^4$

$= c^4 - \frac{8c^4}{16p^4}$

$= \frac{2c^4p^4 - c^4}{2p^4}$ 1 (6)

$\therefore 2xyc^2 = c^4 - 8y^4$.

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⑤ a) i) S is pt $(a\cos\theta, 0)$ or $(ae, 0)$ 1

ii) $ae = a\cos\theta$

$e = \cos\theta$

$\sin\theta = \sqrt{1 - \cos^2\theta}$

$= \sqrt{1 - e^2}$ 1

iii) to find T subst $x = 0$ in Tangent equation.

$y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta} \cdot -a\cos\theta$

$\therefore y = \frac{b\cos^2\theta}{\sin\theta} + b\sin\theta$

$= \frac{b\cos^2\theta + b\sin^2\theta}{\sin\theta}$

$= \frac{b}{\sin\theta}$ 1

T is point $\left(0, \frac{b}{\sin\theta}\right)$

to find K subst $x = 0$ into equation of normal

$y - b\sin\theta = \frac{a\sin\theta}{b\cos\theta}(x - a\cos\theta)$ 1

$\therefore y = b\sin\theta - \frac{a^2\sin\theta}{b}$

$= \frac{b^2\sin\theta - a^2\sin\theta}{b}$

$= \sin\theta \frac{(b^2 - a^2)}{b}$ 1

$\therefore$ K is pt $\left(0, \sin\theta \frac{(b^2 - a^2)}{b}\right)$

(5) i)

(I) S is pt $(a\cos\theta, 0)$ or $(ae, 0)$ 1

(II) $ae = a\cos\theta$

$e = \cos\theta$

$\sin\theta = \sqrt{1-\cos^2\theta}$ 1

$= \sqrt{1-e^2}$

(III) to find T subst. $x=0$ in tangent eqn

$\therefore\ y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}\, x - a\cos\theta$

$y = \frac{b\cos^2\theta}{\sin\theta} + b\sin\theta$

$= \frac{b(\cos^2\theta + \sin^2\theta)}{\sin\theta}$

$= \frac{b}{\sin\theta}$ 1

T is pt $\left(0, \frac{b}{\sin\theta}\right)$ *

to find K subst $x=0$ into equation of the normal

$y - b\sin\theta = \frac{a\sin\theta}{b\cos\theta}(x - a\cos\theta)$ 1

$\therefore\ y = b\sin\theta - \frac{a\sin\theta \cdot a\cos\theta}{b\cos\theta}$

$= b\sin\theta - \frac{a^2\sin\theta}{b}$

$= \frac{b^2\sin\theta - a^2\sin\theta}{b}$

$= \frac{\sin\theta(b^2-a^2)}{b}$ 1

$\therefore$ K is pt $\left(0, \frac{\sin\theta(b^2-a^2)}{b}\right)$ *

M is pt $(0, b\sin\theta)$ 1 *

iv) Area $\triangle TPK$

$= \frac{1}{2} \times TK \times MP$ 1

$= \frac{1}{2} \times \left(\frac{b}{\sin\theta} - \frac{(b^2-a^2)}{b}\right) \times a\cos\theta$

$= \frac{1}{2}\left(\frac{b^2 + a^2\sin^2\theta - b^2\sin^2\theta}{b\sin\theta}\right) a\cos\theta$

$= \frac{(b^2 + a^2(1-e^2) - b^2(1-e^2)) \cdot ae}{2b\sqrt{1-e^2}}$

$= \frac{(b^2 + a^2 - a^2e^2 - b^2 + b^2e^2)\, ae}{2b\sqrt{1-e^2}}$

$= \frac{(a^2(1-e^2) + e^2b^2)\, ae}{2b\sqrt{1-e^2}}$ 1

but $b^2 = a^2(1-e^2)$ 1

$\therefore$ area $= \frac{(b^2 + e^2b^2)\, ae}{2b\sqrt{1-e^2}}$

$= \frac{b^2(e^2+1) \cdot ae}{2b\sqrt{1-e^2}}$

$= \frac{abe(e^2+1)}{2\sqrt{1-e^2}}$ 1 (10)

b) $y = x^3 - px + q$

turn pts when $y' = 0$

i.e $3x^2 - p = 0$ 1

to have 2 turn pts $3x^2 - p = 0$ has 2 solns

i.e $x^2 = \pm\sqrt{\frac{p}{3}}$

$\therefore\ p > 0$ for 2 solns. 1

when $x = +\sqrt{\frac{p}{3}}$

$y = \frac{p}{3}\sqrt{\frac{p}{3}} - p\sqrt{\frac{p}{3}} + q$ 1

$= q - \frac{2p}{3}\sqrt{\frac{p}{3}}$

when $x = -\sqrt{\frac{p}{3}}$

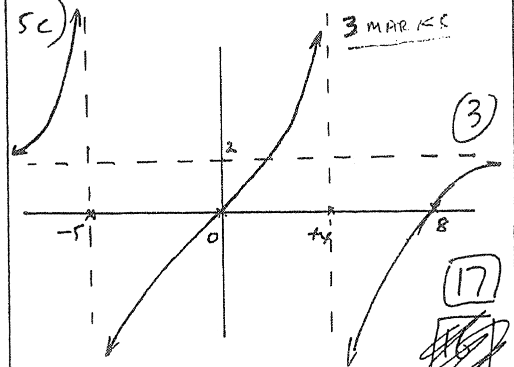
$y = -\frac{p}{3}\sqrt{\frac{p}{3}} - p\sqrt{\frac{p}{3}} + q$ 1

$= q - \frac{4p}{3}\sqrt{\frac{p}{3}}$

as $p > 0$

$q - \frac{2p}{3}\sqrt{\frac{p}{3}} < k < q + \frac{2p}{3}\sqrt{\frac{p}{3}}$ (4)

5c)



(3)

(17)

(16)

(6) a) $P(x) = x^4 + ax^2 + bx$

$x^2+1 = (x+i)(x-i)$ 1

$\therefore\ P(i) = i^4 + ai^2 + bi$

$= 1 - a + bi$ 1

but $P(i) = i + 2$

$\therefore\ 1 - a = 2$

$bi = i$

$\therefore\ a = -1 \quad b = 1$ 1

(3)

b if $P(x) = ax^4 + 4bx + c = 0$ has a double root this is also a root of $P'(x) = 0$

i.e $P'(x) = 4ax^3 + 4b = 0$ 1

$\therefore\ x^3 = \frac{-b}{a}$

$\therefore\ x = \sqrt[3]{\frac{-b}{a}}$ 1

Subst this into $P(x)$

$\therefore\ a\left(\frac{-b}{a}\right)^{\frac{4}{3}} + 4b\left(\frac{-b}{a}\right)^{\frac{1}{3}} + c = 0$ 1

$\therefore\ -\frac{ab}{a}\left(\frac{-b}{a}\right)^{\frac{1}{3}} + 4b\left(\frac{-b}{a}\right)^{\frac{1}{3}} = -c$

$\therefore\ 3b\left(\frac{-b}{a}\right)^{\frac{1}{3}} = -c$

cubing both sides

$27b^3 \times \frac{-b}{a} = -c^3$ 1

$\therefore\ 27b^4 = ac^3$ (4)

c) $\alpha+\beta+\gamma = \frac{-b}{a} = 0$

$\alpha\beta+\alpha\gamma+\beta\gamma = \frac{c}{a} = p$

$\alpha\beta\gamma = \frac{-d}{a} = -q$

i) $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2\Sigma\alpha\beta$ 1

$= 0 - 2p$

$= -2p$ * 1

ii) $(\alpha-1)(\beta-1)(\gamma-1) = \alpha\beta\gamma - \Sigma\alpha\beta + \Sigma\alpha - 1$ 1

$= -q - p - 1$ 1

6c cont

(iii) $x^3 = -px - q$

$\alpha^3 = -p\alpha - q$

$\beta^3 = -p\beta - q$

$\gamma^3 = -p\gamma - q$ 1

$\therefore \alpha^3+\beta^3+\gamma^3 = -p(\alpha+\beta+\gamma) - 3q$

$= -3q$ — * 1

(iv) $x^4 = -px^2 - qx$

$\alpha^4 = -p\alpha^2 - q\alpha$

$\beta^4 = -p\beta^2 - q\beta$ 1

$\gamma^4 = -p\gamma^2 - q\gamma$

$\therefore \alpha^4+\beta^4+\gamma^4 = -p(\alpha^2+\beta^2+\gamma^2) - q(\alpha+\beta+\gamma)$

$= -p \times -2p + 0$ 1 (8)

$= 2p^2$.

15

(7) a) $ma = mg - \frac{mgv}{T}$ 1

$\therefore a = g\left(1 - \frac{v}{T}\right)$

terminal velocity when $a = 0$

ie $\frac{v}{T} = 1$ 1

$\therefore v = T$

$\therefore T$ is terminal velocity. *

when projected upwards with initial velocity u

ie at $t=0$, $x=0$, $v=u$

$ma = -mg - \frac{mgv}{T}$

$a = -g\left(\frac{T+v}{T}\right)$ 1

$\therefore v\frac{dv}{dx} = -g\left(\frac{T+v}{T}\right)$

$\therefore \left(\frac{Tv}{T+v}\right)\frac{dv}{dx} = -g$

$\therefore T\int \frac{v}{T+v}\,dv = \int -g\,dx$ 1

$T\int \frac{v+T-T}{T+v}\,dv = -gx + c$

$T(v - T\ln(T+v)) = -gx + c$ 1

when $v = u$ $x = 0$

$\therefore c = T(u - T\ln(T+u))$

$\therefore x = \frac{T(u - T\ln(T+u) - v + T\ln(T+v))}{g}$

at max height $v = 0$ 1

$\therefore H = T\left(\frac{u - T\ln(T+u) + T\ln T}{g}\right)$

$= \frac{T}{g}\left(u + T\ln\left(\frac{T}{u+T}\right)\right)$ 1

$H = \frac{uT}{g} + \frac{T^2}{g}\ln\left(\frac{T}{u+T}\right)$

(7)

7b) $P(x) = 2x^3 - x^2 - 37x + 36 = 0$

$P(1) = 2 - 1 - 37 + 36 = 0$ 1

1 is a root

by division other roots are $-\frac{9}{2}$, 4.

for $P(x) = 2x^2 - 7x + k$ 1

$P(1)$, $P(\frac{9}{2})$, $P(4) = 0$

$P(1) = 2 - 7 + k = 0$ $\therefore k = 5$ 1

$P(4) = 32 - 28 + k = 0$ $\therefore k = -4$ 1

$P(\frac{-9}{2}) = \frac{81}{2} + \frac{63}{2} + k = 0$ $\therefore k = -72$ 1

(5)

7c) $(1+x)^n\left(1+\frac{1}{x}\right)^n = \left((1+x)\left(1+\frac{1}{x}\right)\right)^n$

$= \left(1+\frac{1}{x}+x+1\right)^n$

$= \left(x+2+\frac{1}{x}\right)^n$

$= \left(\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2\right)^n$ 1

$= \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^{2n}$

$(1+x)^n\left(1+\frac{1}{x}\right)^n = \left({}^nC_0 + {}^nC_1x + {}^nC_2x^2 \ldots {}^nC_nx^n\right) \times \left({}^nC_0 + {}^nC_1x^{-1} + {}^nC_2x^{-2} \ldots {}^nC_nx^{-n}\right)$

Constant term is $= ({}^nC_0)^2 + ({}^nC_1)^2 + ({}^nC_2)^2 \ldots ({}^nC_n)^2$

$= \sum_{r=0}^{n}({}^nC_r)^2$ 1

$\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^{2n} = {}^{2n}C_0(\sqrt{x})^{2n} \ldots {}^{2n}C_r(\sqrt{x})^{2n-r}\left(\frac{1}{\sqrt{x}}\right)^r$

for constant term ~~n-r+r=0~~

$2n - r - r = 0$ 1

$n = r$

ie ${}^{2n}C_n$ (4)

$\therefore \sum_{r=0}^{n}({}^nC_r)^2 = {}^{2n}C_n$

16

(8) $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$ 1

$\therefore \cos 5\theta = Re(\cos\theta + i\sin\theta)^5$

$\cos 5\theta = \cos^5\theta + 10\cos^3\theta\sin^2\theta\, i^2 + 5\cos\theta\sin^4\theta\, i^4$

$= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2$

$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$. * 1

$16x^5 - 20x^3 + 5x = 0$ 1

let $x = \cos\theta$

$\therefore$ substitute equivalent $\cos 5\theta = 0$

$5\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}$ 1

$\theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{\pi}{2}, \frac{7\pi}{10}, \frac{9\pi}{10}$

$\therefore x = \cos\frac{\pi}{10}, \cos\frac{3\pi}{10}, \cos\frac{\pi}{2}, \cos\frac{7\pi}{10}, \cos\frac{9\pi}{10}$ 1

$x = \cos\frac{\pi}{10}, \cos\frac{3\pi}{10}, 0, -\cos\frac{3\pi}{10}, -\cos\frac{\pi}{10}$

solving algebraically

$16x^5 - 20x^3 + 5x = 0$

$x(16x^4 - 20x^2 + 5) = 0$ 1

$\therefore x = 0$ or $x^2 = \frac{20 \mp \sqrt{400-320}}{32}$

$= \frac{5 \mp \sqrt{5}}{8}$

$\therefore x = 0, \mp\sqrt{\frac{5 \mp \sqrt{5}}{8}}$. 1

now ~~$0 < \cos\frac{\pi}{10}$~~

$0 < \cos\frac{3\pi}{10} < \cos\frac{\pi}{10}$

(i) $\therefore \cos\frac{3\pi}{10} = \sqrt{\frac{5-\sqrt{5}}{8}}$ 1

(ii) $\cos\frac{\pi}{10} = \sqrt{\frac{5+\sqrt{5}}{8}}$ 1

(9)

(8) b) $I_n = \int_0^{\frac{\pi}{2}} x^n \sin x \, dx$

let $u = x^n$ $v' = \sin x$

$u' = nx^{n-1}$ $v = -\cos x$

$I_n = \left[-x^n \cos x\right]_0^{\frac{\pi}{2}} + n\int_0^{\frac{\pi}{2}} x^{n-1} \cos x \, dx$

$= 0 + n\int_0^{\frac{\pi}{2}} x^{n-1} \cos x \, dx$ 1

let $u = x^{n-1}$ $v' = \cos x$

$u' = (n-1)x^{n-2}$ $V = \sin x$

$\therefore I_n = n\left[x^{n-1} \sin x\right]_0^{\frac{\pi}{2}} - n(n-1)\int x^{n-2} \sin x \, dx$

$I_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1) I_{n-2}$ 1

$\int_0^{\frac{\pi}{2}} x^4 \sin x \, dx$

$= I_4$

$I_4 = 4\left(\frac{\pi}{2}\right)^3 - 4.3 I_2$

$I_2 = 2\left(\frac{\pi}{2}\right) - 2 I_0$

$I_0 = \int_0^{\frac{\pi}{2}} \sin x \, dx = 1$ 1

$\therefore I_2 = \pi - 2$ 1

$I_4 = 4\left(\frac{\pi}{2}\right)^3 - 12(\pi - 2)$

$= \frac{\pi^3}{2} - 12\pi + 24$

(4)

(8) c)

~~in $\triangle ABD$~~

~~$\cos\theta = \frac{m^2 + \frac{a^2}{4} - a^2}{am}$~~

~~and $\triangle ACD$~~

~~$\cos(\pi - \theta) = \frac{m^2 + \frac{a^2}{4} - b^2}{am} = -\cos\theta$~~

~~$\therefore m^2 + \frac{a^2}{4} - a^2 + m^2 + \frac{a^2}{4} - b^2 = 0$~~

~~$2m^2$~~

8c) in $\triangle ABD$

$\cos\theta = \frac{m^2 + \frac{a^2}{4} - c^2}{2 \cdot \frac{a}{2} \cdot m} = \frac{4m^2 + a^2 - 4c^2}{am}$ 1

in $\triangle ACD$

$-\cos\theta = \frac{m^2 + \frac{a^2}{4} - b^2}{2 \cdot \frac{a}{2} \cdot m} = \frac{(4m^2 + a^2 - 4b^2)}{am}$ 1

$\therefore \frac{4m^2 + a^2 - 4c^2}{am} + \frac{4m^2 + a^2 - 4b^2}{am} = 0$

$8m^2 = 4b^2 + 4c^2 - 2a^2$ 1

$m = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$ (3)

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